Lebesgue Outer Measure

Subject: Measure Theory

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Introduction

- Henri Lebesgue (1875-1941) introduced Lebesgue measure and integration to overcome the limitations of Riemann integration. Riemann integration fails for functions with too many discontinuities. Lebesgue's approach provides a more general and flexible way to define integration. Measure theory provides a rigorous foundation for integration and probability. It extends the concept of length, area, and volume to more complex sets.

Definition of Lebesgue Outer Measure

- For a set A $\subseteq \mathbb{R}$, the Lebesgue outer measure is:

$$m^*(A) = \inf \{ \sum_{n=1}^{\infty} I_n \}$$

where infimum is taken with respect to all countable collections $\{ I_n \}$ of open intervals covering A.

Properties of outer measure

- $m^*(A) \geq 0$ for all sets A.
- $m^*(\emptyset)=0$
- If A and B are two sets with $A \subseteq B$, then $m^*(A) \leq m^*(B)$.
- $m^*(A)=0$ for every singleton set A.
- If A is a set in $\mathbb{R}$ and $x \in \mathbb{R}$, then $m^*(A + x) = m^*(A)$.

Examples of Lebesgue Outer Measure

- If A is countable then $m^*(A) = 0$
- $m^*([0,1]) = 1$
- $m^*(\mathbb{Q} \cap [0,1]) = 0$ (since $\mathbb{Q}$ is countable and can be covered by small intervals).
- Cantor ternary set is measurable and its measure is zero

Measurable Sets

- A set E is Lebesgue measurable if, for all subsets A of $\mathbb{R}$ we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

- A set which is countable union of closed sets is called an F_σ-set.
- A set which is countable intersection of open sets is called anG_δ-set.

Properties

- If the outer measure of a set is zero, then the set is measurable.
- Complement of a measurable set is measurable.
- Union of two measurable set is measurable.
- Every countable set is measurable and its measure is zero.

Borel Sets

- The σ-algebra generated by the family of all open sets in R denoted by $\mathbb{B}$ is called the class of Borel sets in R.
- Every Borel set is measurable.
- Any open interval (a, b) is a Borel set.
- The set of rational numbers $\mathbb{Q}$ is a Borel set.
- Any countable subset of $\mathbb{R}$ is a Borel set.

Conclusion

- Lebesgue outer measure provides a rigorous foundation for measure theory, leading to powerful applications in analysis and probability.

References & Acknowledgments

- H. Lebesgue, 'Intégrale, longueur, aire'
- W. Rudin, 'Real and Complex Analysis'
- G. B. Folland, 'Real Analysis'